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# Some Interpolation Theorems for Polynomials

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Let $E = \| e_{ij} \|_{j=0,\ldots,n-1}^{i=1,\ldots,k}$ be a given $n$-incidence matrix and suppose knots $x_1 < x_2 < \cdots < x_k$ are given. This paper studies the following problem related to the matrix $E$: if $p$ is an integer, $1 \leqslant p \leqslant n-1$, and $g(x) \in C^{(n-p)}[x_1, x_k]$, does there exist a function $f(x)$ satisfying

(i) $f^{(j)}(x_i) = 0$ when $e_{ij} = 1$ and

(ii) $f^{(p)}(x) \equiv g(x)$?

Certain functions $W_1(t),\ldots, W_{n-1}(t)$ which do not depend on $g$ are constructed with the result that for almost all choices of the knots $x_i$ a solution exists if

$$\int_{x_1}^{x_k} g^{(q)}(t)\, W_q^{(n-q)}(t)\, dt = 0 \qquad \text{for} \qquad q = p,\ldots, n-1.$$

This result is applied to the nonhomogenous problem where data $y_{ij}$ is prescribed and (i) is replaced with $f^{(j)}(x_i) = y_{ij}$. Also, the concept of a simple matrix is introduced, and some results on the relation between poised and simple matrices are given.

## INTRODUCTION

Some interpolation problems which arise from the study of Hermite Birkhoff systems are examined in this paper. The main problem studied here is the following: Let $E = \| e_{ij} \|_{j=0,\ldots,n-1}^{i=1,\ldots,k}$ be an $n$-incidence matrix. Suppose $x_1, \ldots, x_k$ are given, along with an integer $p$, $1 \leqslant p \leqslant n-1$, and a function $g \in C^{(n-p)}$. When does there exist a function $f(x)$ with the properties:

(i) $f^{(j)}(x_i) = 0$ whenever $e_{ij} = 1$, and

(ii) $f^{(p)}(x) \equiv g(x)$?

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This problem is important for the study of properties of functions $f(x)$ which satisfy ($i$). For example, the question of when $(i) \Rightarrow f^{(n-1)}(x)$ has a zero leads immediately to the above problem for $p = n - 1$.

The necessary background and machinery are developed in the first two sections. Section 3 contains a discussion of the above problem and its complete solution when $E$ is poised at $\mathbf{x}$ (see Section 1). Section 4 presents some applications and examples. The applications deal with the problem (defined in the paper) of simple matrices and with the problem of interpolation of nonhomogenous data. Section 5 presents a proof of Theorem 3.3.

## 1. Preliminaries

Let $E$ denote the $n$-incidence matrix $\| e_{ij} \|_{j=0,\ldots,n-1}^{i=1,\ldots,k}$ where each $e_{ij}$ is either zero or one and the sum of the entries is $n$ ($\sum_{i=1}^{k} \sum_{j=0}^{n-1} e_{ij} = n$). For a given vector $\mathbf{x} = (x_1, \ldots, x_k) \in R^k$ with components that satisfy $x_1 < x_2 < \cdots < x_k$ define the class of functions $Z(E, \mathbf{x})$ by

$$f \in Z(E, \mathbf{x}) \quad \text{iff} \quad f^{(j)}(x_i) = 0 \quad \text{when} \quad e_{ij} = 1.$$

Let $\Pi_{n-1}$ be the class of polynomials of degree less than or equal to $n - 1$.

Definition 1.1. $E$ is *poised at* $\mathbf{x}$ if $Z(E, \mathbf{x}) \cap \Pi_{n-1} = 0$ (the zero polynomial); $E$ is *order-poised* if it is poised at all $\mathbf{x}$ satisfying $x_1 < x_2 < \cdots < x_k$; $E$ is *simple at* $\mathbf{x}$ if $f \in Z(E, \mathbf{x}) \Rightarrow$ each of the functions $f, f', \ldots, f^{(n-1)}$ vanishes at least once on the interval $[x_1, x_k]$.

The concept of a simple matrix is new. It is to be regarded as a generalization of Rolle's Theorem. Note that Rolle's Theorem is precisely the statement that the matrix $\left\| \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right\|$ is simple at all $\mathbf{x} = (x_1, x_2)$. Except for Theorem 1.3, simple matrices do not appear until Section 4.

As examples, consider the matrices

$$E_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix} \quad \text{and} \quad E_2 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}.$$

$E_1$ is order-poised and simple. In fact, let $f \in Z(E_1, \mathbf{x})$. Then Rolle's Theorem says that $f'$ has an odd order zero in the open interval $(x_1, x_3)$. Thus, either $f'''(x_2) = 0$ or there is a point $\alpha \neq x_2$ at which $f'(\alpha) = 0$. In the second case Rolle's Theorem can be applied twice more to yield a point $\beta \in (x_1, x_3)$ for which $f'''(\beta) = 0$. In both cases $f$ satisfies the condition for being simple since $f, f', f'', f'''$ all vanish at least once. Also, if $f \not\equiv 0$, then $f \notin \pi_3$ and $E_1$ is order-poised.

$E_2$ is neither order-poised nor simple. Take $x_1 = 0$, $x_3 = 1$. If $x_2 = 1/2$, $E_2$ is not poised by selecting $f(x) = x(x - 1)$. For $x_2 \neq 1/2$, $E_2$ is poised. Choosing $f(x) = e^{\alpha x} - x(e^{\alpha} - 1) - 1$ where $\alpha e^{\alpha x_2} - e^{\alpha} + 1 = 0$ gives a function which satisfies $f(0) = f(1) = f'(x_2) = 0$ and $f''(x) = \alpha^2 e^{\alpha x} \neq 0$ for every choice of $x$. Thus, $E_2$ is not simple at any $\mathbf{x} = (x_1, x_2, x_3)$ with $x_1 < x_2 < x_3$.

The proof of Theorem 3.3 requires the concept of an unconditionally poised matrix.

DEFINITION 1.2. $E$ is *unconditionally poised if*, for given distinct complex numbers $z_1, \ldots, z_k$ the condition that $p(z) \in \Pi_{n-1}$ and $p^{(j)}(z_i) = 0$ if $e_{ij} = 1$ implies $p(z) \equiv 0$.

The example $E_1$ is not unconditionally poised. Choose $x_1$, $x_3$ to be distinct cube roots of unity and $x_2 = 0$. Then $p(x) = x^3 - 1$ shows that $E_1$ is not poised at $(x_1, x_2, x_3)$.

Let $m_j = \sum_{i=1}^{k} e_{ij}$; $M_j = \sum_{p=0}^{j} m_p$. Note that $M_j = M_{j-1} + m_j$ and $M_{n-1} = n$.

DEFINITION 1.3. $E$ satisfies the *Polya conditions* (PC) if $M_j \geqslant j + 1$ for $j = 0, \ldots, n - 1$. If equality holds for some $j$ then $E$ may be written as $E = E' \oplus E''$ where $E'$ consists of columns 0 thru $j$ of $E$ and $E''$ consists of the remaining columns. $E$ satisfies the *strong Polya conditions* (SPC) if $M_j \geqslant j + 2$ for $j = 0, \ldots, n - 2$.

Note that matrices satisfying (SPC) do not admit decompositions of the form $E' \oplus E''$. The following theorems characterize matrices that are poised for some $\mathbf{x}$ and matrices that are unconditionally poised. The reader is referred to [3] for proofs.

THEOREM 1.1. (i) *There exists a vector* $\mathbf{x}$ *at which $E$ is poised iff $E$ satisfies* (PC). *In this case the set of vectors* $\mathbf{x}$ *at which $E$ fails to be poised is nowhere dense in* $R^k$. (ii) *If* $E = E' \oplus E''$, *then it is poised at* $\mathbf{x}$ *iff* $E'$, $E''$ *are poised at* $\mathbf{x}$.

THEOREM 1.2. *Let $E$ satisfy* (SPC). *$E$ is unconditionally poised iff* $k = 2$ *or $E$ is a Hermite matrix (i.e., if* $e_{ij} = 1$ *then* $e_{ij'} = 1$ *for each* $j' \leqslant j$).

*Remark* 1.1. If $E$ satisfies (PC), $E$ is poised a.e. The a.e. restriction appears throughout this paper (see the theorems of Sections 3 and 5) and cannot generally be removed.

The following theorem relates poised matrices and simple matrices.

THEOREM 1.3. *$E$ simple at* $\mathbf{x} \Rightarrow$ *$E$ poised at* $\mathbf{x}$.

*Proof.* Let $E$ be simple and $p(x)$ be a polynomial of exact degree $m \geqslant 0$. $p^{(m)}(x) \equiv$ const. Thus, if $p(x) \in Z(E, \mathbf{x})$, $m \geqslant n$ and $E$ is poised. ∎

The converse of this theorem is not true in general as will be shown in Section 4. The class of conservative matrices (see [1, 4, 5] and also Section 4) provides a large collection of simple matrices.

If $E$ is poised at $\mathbf{x}$, then the linear system

$$a_{n-1} \frac{x_i^{n-1-j}}{(n-1-j)!} + \cdots + a_{j+1} x_i + a_j = y_{ij}\,; \qquad e_{ij} = 1$$

has a unique solution for each choice of the values $y_{ij}$. Let $V$ be the coefficient matrix of (1.1). Thus,

$$V = \left( D^j \frac{x^{n-1-p}}{(n-1-p)!} \bigg|_{x=x_i} \right)_{p=0,\ldots,n-1}^{e_{ij}=1}$$

with the index pairs $(i, j)$ for which $e_{ij} = 1$ forming the rows of $V$. $V$ will be called the Vandermonde (VdM) matrix of $E$, $\Delta = \det V$ will be called the VdM determinant of $E$. $E$ is poised at $\mathbf{x}$ iff $\Delta \neq 0$.

As an example, consider the matrix

$$E_1 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}.$$

In this case,

$$V = \begin{pmatrix} x_1^2/2 & x_1 & 1 \\ x_2 & 1 & 0 \\ x_3^2/2 & x_3 & 1 \end{pmatrix}.$$

No convention is made regarding the ordering of the rows of $V$ since no formal matrix algebra is ever performed on $V$. Thus, any permutation of the rows of $V$ will be a valid representation in what follows.

In the rest of the paper the notation will be simpler if the following conventions are adopted.

1. $\mathbf{x}$ will always represent a vector of the form $\mathbf{x} = (x_1, \ldots, x_k)$ with $0 \leqslant x_1 < \cdots < x_k \leqslant 1$;

2. *If* $\Gamma_{ij}$ are objects (numbers, functions, etc.) corresponding to $e_{ij} = 1$, then $\sum \Gamma_{ij}$ will denote the sum taken over all index pairs $(i, j)$ with $e_{ij} = 1$. Similarly, $\sum_{(i,j) \Rightarrow S} \Gamma_{ij}$ will denote the sum taken over all index pairs $(i, j)$ which satisfy the statement $S$ and satisfy

$e_{ij} = 1$. For example $\sum_{j\leqslant 3} \Gamma_{ij}$ means the sum over all pairs $(i, j)$ with $e_{ij} = 1$ and $j \leqslant 3$.

3. $\int_0^1 f(t)\, g(t)\, dt \equiv \langle f, g\rangle$.

## 2. The Peano Kernel $K(x, t)$ and its Properties

Suppose $E$ is poised at $\mathbf{x}$. Let $L_{ij}(x)$ denote the unique elements of $\Pi_{n-1}$ which satisfy

$$L_{ij}^{(j')}(x_{i'}) = \delta_{(i,j)(i',j')} \qquad \text{where} \quad e_{ij} = e_{i'j'} = 1.$$

$Rf$ is the linear operator defined by

$$Rf(x) = f(x) - \sum L_{ij}(x)\, f^{(j)}(x_i). \tag{2.2}$$

Since $Rf \equiv 0$ for $f \in \Pi_{n-1}$, Peano's Theorem characterizes $R$ on the class $C^{(n)}[0, 1]$ as

$$Rf(x) = \int_0^1 f^{(n)}(t)\, K(x, t)\, dt \tag{2.3}$$

where

$$K(x, t) = \frac{(x - t)_+^{n-1}}{(n-1)!} - \sum L_{ij}(x) \frac{(x_i - t)_+^{n-1-j}}{(n-1-j)!}. \tag{2.4}$$

(The function $(y - z)_+^p$ is defined by

$$(y - z)_+^p = \begin{cases} (y - z)^p & \text{if} \quad y \geqslant z, \\ 0 & \text{if} \quad y < z). \end{cases}$$

THEOREM 2.1. (i) $f \in Z(E, \mathbf{x}) \cap C^{(n)}[0, 1]$ *iff* $f(x) \equiv Rf$.

(ii) *If* $f \in C[0, 1]$, *then* $g(x) = \int_0^1 f(t)\, K(x, t)\, dt \in Z(E, \mathbf{x}) \cap C^{(n)}[0, 1]$.

*Proof.* (i) is trivial. To prove (ii) observe that

$$g^{(j)}(x_i) = \int_0^1 f(t) \left(\frac{\partial^j}{\partial x^j} K(x, t)\Big|_{x=x_i}\right) dt$$

and

$$\frac{\partial^j}{\partial x^j} K(x, t)\Big|_{x=x_i} \equiv \frac{(x_i - t)_+^{n-1-j}}{(n-1-j)!} - \frac{(x_i - t)_+^{n-1-j}}{(n-1-j)!} \equiv 0 \qquad \text{if} \quad e_{ij} = 1.$$

Thus, $g \in Z(E, \mathbf{x}) \cap C^{(n)}[0, 1]$. ∎

For $x \in [x_1, x_k]$, $K(x, t)$ has the following properties.

$$K(x, t) \equiv 0 \quad \text{for} \quad t < x_1 \quad \text{and} \quad t > x_k . \tag{2.5}$$

This follows for $t > x_k$ by the definition of $(x - t)_+^k/k!$. For fixed $t < x_1$, $K(x, t) \in \Pi_{n-1}$ and $K(x, t) \in Z(E, \mathbf{x})$. Since $E$ is poised at $\mathbf{x}$, $K(x, t) \equiv 0$.

$$K(x, t) = \frac{(x - t)_+^{n-1}}{(n - 1)!} - \frac{1}{\Delta} \sum_{p=0}^{n-1} W_p(t) \frac{x^p}{p!} \tag{2.6}$$

where $\Delta$ is the VdM determinant of $E$,

$$W_p(t) = \sum \Delta_{ij}^{(p)} \frac{(x_i - t)_+^{n-1-j}}{(n - 1 - j)!}$$

and $\Delta_{ij}^{(p)}$ is the cofactor in $\Delta$ of the element $x_i^{p-j}/(p - j)!$. The representation follows by observing that $L_{ij}(x) = 1/\Delta \sum_{p=0}^{n-1} \Delta_{ij}^{(p)}(x^p/p!)$ and then rearranging the expression for $K(x, t)$.

$$\begin{array}{lll} \text{(i)} & W_p(t) \equiv 0 & \text{for} \quad t > x_k \\ \text{(ii)} & W_p(t) \in \Pi_{n-1} & \text{for} \quad x_{i-1} < t < x_i \end{array} \tag{2.7}$$

(iii) $W_p^{(j)}(t)$ is discontinuous at $t = x_i$ iff $e_{i,n-1-j} = 1$ and $\Delta_{i,n-1-j}^{(p)} \neq 0$.

(iv) $W_p(t) \equiv (-1)^{n-1-p} \dfrac{t^{n-1-p}}{(n - 1 - p)!} \Delta \quad$ for $\quad t < x_1$.

(v) $W_p^{(n-p)}(t) \equiv 0 \quad$ for $\quad t < x_1$.

Statement (iv) follows from using the fact that $K(x, t) \equiv 0$, $t < x_1$ and equating powers of $x$.

The functions $W_p(t)$ play a crucial role in what follows.

LEMMA 2.1. *Let $f \in C^n[0, 1]$. For $p = 1,..., n - 1$,*

$$\langle f^{(p)}, W_p^{(n-p)} \rangle = (-1)^{n-p} \sum_{j \leq p-1} \Delta_{ij}^{(p)} f^{(j)}(x_i)$$

*while*

$$\int_0^1 f(t)\, dW_0^{(n-1)}(t) = (-1)^n \sum_{j=0} \Delta_{ij}^{(p)} f^{(j)}(x_i).$$

*Proof.*

$$W_p^{(n-p)}(t) = (-1)^{n-p} \sum_{j \leq p-1} \Delta_{ij}^{(p)} \frac{(x_i - t)_+^{p-1-j}}{(p - 1 - j)!}$$

If $W_p^{(n-p)}(t)$ is continuous at $t = x_1$ or $x_k$, its value is zero. This follows from (2.7) (i) and (v).

$$\langle f^{(p)}, W_p^{(n-p)} \rangle = \sum_{i=1}^{k-1} \int_{x_i}^{x_{i+1}} f^{(p)}(t)\, W_p^{(n-p)}(t)\, dt.$$

Repeated integration by parts yields the formula

$$\int_{x_i}^{x_{i+1}} f^{(p)}(t)\, W_p^{(n-p)}(t)\, dt = \sum_{j=0}^{p-1} (-1)^{p-1-j} f^{(j)}(x)\, W_p^{(n-j-1)}(x) \Big|_{x_i}^{x_{i+1}}$$

Hence,

$$\langle f^{(p)}, W_p^{(n-p)} \rangle = \sum_{i=0}^{k-1} \sum_{j=0}^{p-1} (-1)^{p-1-j} f^{(j)}(x)\, W_p^{(n-j-1)}(x) \Big|_{x_i}^{x_{i+1}}.$$

Contribution to the sum only occurs at points $x_i$ where $W_p^{(n-j-1)}(x)$ is discontinuous. At such points the contribution is equal to $f^{(j)}(x_i)\, \Delta_{ij}^{(p)}(-1)^{p-j-1}$ $(-1)^{n-j-1}$ which establishes the lemma for $p = 1,..., n-1$. For $p = 0$,

$$\int_0^1 f(t)\, dW_0^{(n-1)}(t) = (-1)^n \sum_{j=0} \Delta_{ij}^{(0)} f(x_i)$$

since $W_0^{(n-1)}(t) \equiv (-1)^n \sum_{j=0} \Delta_{ij}^{(0)}(x_i - t)_+^0$. ∎

*Remark* 2.1. Setting $p = n - 1$ yields

$$\langle f^{(n-1)}, W'_{n-1} \rangle = \sum_{j \leqslant n-2} f^{(j)}(x_i)\, \Delta_{ij}^{(n-1)}.$$

This formula is due originally to G. D. Birkhoff [2].

## 3. Interpolation Theorems

### 1. *Introduction*

This section contains the main results of the paper. $E$ is assumed throughout to be poised at $\mathbf{x}$ and to satisfy the *strong Polyá conditions* $M_j \geqslant j + 2$ for $j = 0,..., n-2$ (see Definition 1.3). The problem considered is the following.

(*) Let $p$ be a fixed integer, $1 \leqslant p \leqslant n-1$ and $g \in C^{(n-p)}[0, 1]$. When does there exist $f \in Z(E, \mathbf{x})$ for which $f^{(p)} \equiv g$?

The case $p = n - 1$ has been considered by Birkhoff [2].

Notice that the assumption that $E$ is poised at $\mathbf{x}$ implies uniqueness of any solution to (*). In fact if $f_1$ and $f_2$ are two such solutions then

$f_1 - f_2 \in Z(E, \mathbf{x})$ is a polynomial of degree at most $p - 1$. Thus, $f_1 - f_2 \equiv 0$. The problem (*) also has meaning for $p \geqslant n$ and the solution is easily obtained. Let $\hat{g}(x)$ be a $p$-fold integral of $g$. There is a unique $p(x) \in \Pi_{n-1}$ satisfying $\hat{g}^{(j)}(x_i) + p^{(j)}(x_i) = 0$ for $e_{ij} = 1$. Let $f(x) = \hat{g}(x) + p(x)$. Then $f \in Z(E, \mathbf{x})$ and $f^{(p)} \equiv g$. For $p = n$ the solution is unique while for $p > n$, $f^{(n)} \equiv \hat{g}^{(n)}$ and $\hat{g}^{(n)}$ contains $p - n$ arbitrary parameters and the solution exists and is not unique. The problem (*) with $1 \leqslant p \leqslant n - 1$ has a solution when, and only when, the polynomial $p(x)$ obtained above has degree less than $p$.

Cramer's rule gives $p(x)$ as

$$p(x) = \sum_{q=0}^{n-1} A_q \frac{x^q}{q!} \qquad \text{where} \quad A_q = -\frac{1}{\Delta} \sum \hat{g}^{(j)}(x_i)\, \Delta_{ij}^{(q)}.$$

Thus, a necessary and sufficient condition for (*) to have a solution is the vanishing of the quantities $\sum \hat{g}^{(j)}(x_i)\, \Delta_{ij}^{(q)}$ for $q = p,\dots, n - 1$. Clearly, a necessary condition for a solution is that $g^{(j-p)}(x_i) = 0$ for each $e_{ij} = 1$ with $j \geqslant p$. This leads to the following.

THEOREM 3.1. *Problem* (*) *has a solution iff*

$$g^{(j-p)}(x_i) = 0 \qquad \text{if} \quad e_{ij} = 1, \qquad j \geqslant p \tag{3.1}$$

*and*

$$\sum_{j \leqslant q-1} \hat{g}^{(j)}(x_i)\, \Delta_{ij}^{(q)} = (-1)^{n-q} \langle \hat{g}^{(q)}, W_q^{(n-q)} \rangle$$

$$= (-1)^{n-q} \langle g^{(q-p)}, W_q^{(n-q)} \rangle = 0 \tag{3.2}$$

*for* $q = p,\dots, n - 1$.

The rest of this section shows the rather surprising fact that (3.2) is not only a necessary condition, but for almost every $\mathbf{x}$ it is a sufficient condition. The proof begins by studying a certain linear system that arises from the fact that $W_q^{(n-q)}(t) \equiv 0$ for $t < x_1$. Conditions are given under which the solutions which are furnished by the coefficients of $W_q^{(n-q)}$ span the null space of the linear system (Theorem 3.2). The problem of when (3.2) $\Rightarrow$ (3.1) is then attacked. It is reduced by means of Theorem 3.2 to showing that another linear system is nonsingular.

## 2. *A Linear System*

Consider the functions $W_p^{(n-p)}$ for $q = p,\dots, n - 1$. By property (2.7v) these functions all vanish identically for $t < x_1$.

Thus,

$$0 \equiv W_q^{(n-p)}(t) = \sum_{j \leqslant p-1} \Delta_{ij}^{(q)}((x_i - t)_+^{p-j-1}/(p-j-1)!) \qquad \text{for} \quad t < x_1 .$$

Equating powers of $t$ yields

$$\frac{(-1)^r}{r!} \sum_{j \leqslant p-1} \Delta_{ij}^{(q)} \frac{x_i^{p-1-j-r}}{(p-1-j-r)!} = 0 \qquad \text{for} \quad r = 0,..., p-1 \tag{3.3}$$

and $q = p,..., n-1$. Hence, the vectors $V_q^T = (\Delta_{ij}^{(q)})_{j \leqslant p-1}^T$ provide $n-p$ solutions to the linear system

$$A\mathbf{v} = 0$$

where $A$ is the $p \times M_{p-1}$ matrix

$$A = \left(\frac{x_i^{p-1-j-r}}{(p-1-j-r)!}\right)_{e_{ij}=1; j \leqslant p-1}^{r=0,...,p-1} .$$

(The index pairs $(i, j)$ with $e_{ij} = 1, j \leqslant p-1$ determine the columns of $A$.)

Now $A^T$ is the VdM matrix of the truncated matrix $E^{(p)} = \| e_{ij} \|_{j \leqslant p-1}$. Since $E$ is poised at $\mathbf{x}$ there is no nontrivial $p(x) \in \Pi_{p-1}$ satisfying $p^{(j)}(x_i) = 0$ for all $e_{ij} = 1, j \leqslant p-1$. Hence, det $A^T$ has a nonzero $p \times p$ minor and the dimension of the null space of (3.3) is at most $M_{p-1} - p \leqslant n - p$. Thus, there are more than enough vectors $V_q^T$ to span the null space of (3.3). The problem now is to produce $M_{p-1} - p$ of them that are linearly independent.

Let $V$ be the VdM matrix of $E$ (see Section 1, formula 1.1). Represent $r \times r$ minors of $V$ (and similarly $V^{-1}$) by $V\binom{Z_1,...,Z_r}{q_1,...,q_r}$ where the $Z_i$'s are index pairs $(s, t)$ with $e_{s,t} = 1$. Note that for fixed $p$ the vector $(1/\Delta)\, V_q^T$ consists of the first $M_{p-1}$ components of the $q$-th row of $V^{-1}$.

THEOREM 3.2. *Let $E$ satisfy* (SPC) *and set $L_{p-1} = M_{p-1} - p$. For almost every choice of* $\mathbf{x}$ *the following two statements hold simultaneously:*

(i) *the vectors $V_p^T,..., V_{p-1+L_{p-1}}^T$ span the null space of* (3.2) *and*

(ii) *if $m_p \geqslant 1$ then for each $l = 1,..., m_p$ there are constants $a_{p+l} \neq 0$, $b_q^{(l)}$ for which $V_p^T = a_{p+l} V_{p+l}^T + \sum_{q=p+m_p+1}^{M_p-1} b_q^{(l)} V_q^T$.*

*Proof.* Consider the vectors $\{V_{q_s}\}_{s=1}^{L_{p-1}}$ where the $q_s$'s form an increasing sequence of integers with $q_1 \geqslant p$. These vectors are independent iff there are $L_{p-1}$ columns $Z_1 ,..., Z_{L_{p-1}}$ among the first $M_{p-1}$ columns of $V^{-1}$ for which

$$V^{-1}\begin{pmatrix} q_1 ,..., q_{L_{p-1}} \\ Z_1 ,..., Z_{L_{p-1}} \end{pmatrix} \neq 0.$$

This is nonzero iff

$$V\begin{pmatrix} Z_1',\ldots, Z'_{n-L_{p-1}} \\ \\ q_1',\ldots, q'_{n-L_{p-q}} \end{pmatrix} \neq 0$$

where both $\{Z\} \cup \{Z'\}$ and $\{q\} \cup \{q'\}$ are complete enumerations of the rows and columns of $V$.

Construct an $n$-incidence matrix $E^*$ as follows:

$$E^* = \| e_{ij}^* \|_{j=0,\ldots,n-1}^{i=1,\ldots,k+1}$$

where

$$e_{ij}^* = \begin{cases} 1 & \text{if } \ i = k+1, \ \ j = q_s \ \text{ for } \ 1 \leqslant s \leqslant L_{p-1} \\ & \text{or } \ (i,j) = Z_s' \ \text{ for } \ 1 \leqslant s \leqslant n - L_{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $E^*$ is obtained from $E$ by first adjoining a $(k+1)$ row with $L_{p-1}$ ones corresponding to the indices $q_s$ and then changing the index pairs $Z_s$ from one to zero.

Let $\Delta^*$ be the VdM determinant of $E^*$. The utility of $E^*$ lies in the fact that

$$V\begin{pmatrix} Z_1',\ldots, Z'_{n-L_{p-1}} \\ \\ q_1',\ldots, q'_{n-L_{p-1}} \end{pmatrix} = \pm\Delta^* |_{x_{k+1}=0} .$$

In order to establish this fact observe that at $x_{k+1} = 0$ the rows of $\Delta^*$ which correspond to index pairs $(k+1, q_s)$ consist entirely of zeros except for a single one in the $q_s$ position. Thus, these rows and columns may be deleted from $\Delta^*$ and the result only affects the sign of the determinant. However, this can also be obtained from the determinant $\Delta$ by removing the rows corresponding to index pairs $z_s$ and columns corresponding to $q_s$.

Applying Theorem 1.1(i) yields the following.

LEMMA 3.1. *The vectors* $\{V_{q_s}\}_{i=1}^{L_{p-1}}$ *are independent for almost every* $\mathbf{x}$ *iff* $M_j^* \geqslant j+1$ *for each* $j = 0,\ldots, n-1$; *i.e.,* $E^*$ *satisfies* (PC).

To complete the proof of Theorem 3.2 it remains to show that index pairs $\{Z_s\}$ and indices $\{q_s\}$ can be selected so that $E^*$ satisfies (PC) and so that statements (i) and (ii) hold. Since $E$ is poised at $\mathbf{x}$ there is a $p$-incidence submatrix $E'$ of the matrix $E^{(p)} = \| e_{ij} \|_{i \leqslant p}$ which is $p$-poised at $\mathbf{x}$. Let the remaining $M_{p-1} - p = L_{p-1}$ ones of $E^{(p)}$ form the set $Z_1,\ldots, Z_{L_{p-1}}$. Now $E^* = E' \oplus E''$ where $E''$ is the matrix $\| e_{ij} \|_{j \geqslant p}$ with a column with ones in the $q_s - p$ positions, $s = 1,..., L_{p-1}$ adjoined to it. $E'$ is poised at $\mathbf{x}$. If $q_s = p + (s - 1)$, $s = 1,..., L_{p-1}$, then $E''$ satisfies (PC) and is poised almost everywhere. By Theorem 1.1(ii) $E^*$ is also poised almost everywhere and statement (i) of Theorem 3.2 holds.

Statement (ii) still has to be shown. In order that the matrix $E''$ defined above satisfy (PC) it is sufficient to choose $q_s = p + m_p + s - 1$ for $s = 2,..., L_{p-1}$ and $q_1$ to be any of the numbers $p,..., p + m_p$. Thus, for almost every $\mathbf{x}$ the vectors $V_{p+l}$ and $\{V_q\}_{q=p+m_p+1}^{M_p-1}$ for $l = 0,..., m_p$ span the solution set of (3.3). For $l \neq 0$ this gives the representation (ii) of the vector $V_p$. Also, $a_{p+l} \neq 0$ since, if it were zero, then the vectors $V_p$ and $\{V_q\}_{q=p+m_p+1}^{M_q-1}$ would not be independent. This contradicts the fact that $E^*$ is poised at $\mathbf{x}$. ▮

The "almost every" condition of Theorem 3.1 can not be removed in general. Thus, if

$$E = \left\| \begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{matrix} \right\|$$

and $p = 2$, then

$$E^* = \left\| \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \right\| \oplus \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \right\|$$

and the second matrix is not unconditionally poised.

The following Corollary gives some conditions under which the almost every qualification may be dropped.

COROLLARY 3.1. *Let $l$ be an integer for which the matrix $\| e_{ij} \|_{j \geqslant l}$ is hermitian; i.e., $e_{i,j} = 1$ and $j \geqslant l \Rightarrow e_{i,j'} = 1$ for each $j' = l,..., j$. Then the conclusions of Theorem 3.2 hold without reservation for each $p = l,..., n - 1$.*

*Proof.* The Corollary is true iff the matrix $E''$ constructed in the proof is order-poised. But under the stated conditions $E''$ is quasihermite. Hence it is order poised [6]. ▮

3. (3.2) $\Rightarrow$ (3.1)

The following theorem and its corollary will be needed here. Its proof is deferred to Section 5.

THEOREM 3.3. *As a function of* $\mathbf{x}$, $\Delta \equiv \sum_{j=p} \Delta_{ij}^{(p)}$ *iff* $M_{p-1} = p$. *If* $E$ *satisfies* (SPC) *then* $\Delta = \sum_{j=p} \Delta_{ij}^{(p)}$ *only if* $p = 0$.

COROLLARY 3.2. *If E satisfies* (SPC), *then almost everywhere* $\Delta \neq \sum_{j=p} \Delta_{ij}^{(p)}$ *for* $p \geqslant 1$.

The machinary has now been developed for proving that statement (3.2) of Theorem 3.1 implies statement (3.1).

THEOREM 3.4. *Let E satisfy* (SPC) *and the integer p*, $1 \leqslant p \leqslant n-1$ *and* $g \in C^{(n-p)}[0, 1]$ *be given. For almost all* $\mathbf{x}$, $\langle g^{(q-p)}, W_q^{(n-q)} \rangle = 0$ *for* $q = p, \ldots, n-1 \Rightarrow g^{(j-p)}(x_i) = 0$ *for* $e_{ij} = 1$ *and* $j \geqslant p$.

*Proof.* Assume the implication holds on the range $p+1, \ldots, n-1$. The case for $p = n-1$ follows from the fact that $\{i: e_{i,n-1} = 1\} = \varnothing$ when $E$ satisfies (SPC). Thus, it may be assumed that $g^{(j-p)}(x_i) = 0$ for $e_{ij} = 1$, $j \geqslant p+1$ and it only remains to show that $g(x_i) = 0$ whenever $e_{ip} = 1$. If $m_p = 0$, this is trivially satisfied. Thus, it may be assumed that $m_p > 0$.

Lemma 2.1 gives

$$\langle g, W_p^{(n-p)} \rangle = \sum_{j \leqslant p-1} \hat{g}^{(j)}(x_i)\, \Delta_{ij}^{(p)} = 0 \tag{3.4}$$

where $\hat{g}$ is a $p$-fold integral of $g$. Furthermore,

$$\langle g, W_q^{(n-q)} \rangle = \sum_{j \leqslant p} \hat{g}^{(j)}(x_i)\, \Delta_{ij}^{(q)} = 0, \qquad q = p+1, \ldots, n-1, \tag{3.5}$$

again by Lemma 2.1 and the inductive hypothesis. Thus, for arbitrary constants $a_{p+l}$, $b_q^{(l)}$ (3.5) yields

$$\sum_{j \leqslant p} \hat{g}^{(j)}(x_i) \left\{ a_{p+l} \Delta_{ij}^{(p+l)} + \sum_{q=p+m_p+1}^{M_p-1} b_q^{(l)} \Delta_{ij}^{(q)} \right\} = 0 \tag{3.6}$$

for $l \geqslant 1$. According to Theorem 3.2 the constants $a_{p+l} \neq 0$, $b_q^{(l)}$ can be chosen so that the quantity inside the curly brackets reduces to $\Delta_{ij}^{(p)}$ for $j \leqslant p-1$. But then relation (3.4) says these quantities can be deleted from the sum. Thus, (3.6) reduces to

$$\sum_{i \in \Lambda_p} \hat{g}^{(p)}(x_i) \left\{ a_{p+l} \Delta_{ip}^{(p+l)} + \sum_{q=p+m_p+1}^{M_p-1} b_q^{(l)} \Delta_{ip}^{(q)} \right\} = 0 \tag{3.7}$$

for $l = 1, \ldots, m_p$, $\Lambda_p = \{i: e_{ip} = 1\}$.

Let $c_{l,i}$ be the quantity in the curly brackets of (3.7) and $\mathbf{c}_l = (c_{l,i})_{i \in \Lambda_p}$. There are $m_p$ such vectors each of length $m_p$. If they are independent then (3.7) yields the desired result $0 = \hat{g}^{(p)}(x_i) = g(x_i)$ whenever $e_{ip} = 1$.

Suppose they are dependent. Then there are constants $d_l$ not all zero for which $\sum_{l=1}^{m_p} d_l \mathbf{c}_l = 0$. This gives

$$\sum_{l=1}^{m_p} d_l \left\{ a_{p+l} \Delta_{ip}^{(p+l)} + \sum_{q=p+m_p+1}^{M_p-1} b_q^{(l)} \Delta_{ip}^{(q)} \right\} = 0 \tag{3.8}$$

for each $i \in \Lambda_p$. Let $V_q{}^* = (\Delta_{ij}^{(q)})_{j \leqslant p}$ and $\hat{V}_p$ be $V_p$ with $m_p$ zeros joined on to it. Relation (3.8) and the way the constants $a_{p+l}$, $b_q^{(l)}$ were chosen yields

$$\sum_{l=1}^{m_p} d_l \left\{ a_{p+l} V_{p+l}^* + \sum_{q=p+m_p+1}^{M_p-1} b_q^{(l)} V_q{}^* \right\} = \left( \sum_{l=1}^{m_p} d_l \right) V_p . \tag{3.9}$$

Let $f_l = d_l a_{p+l}$, $e_q = \sum_{l=1}^{m_p} d_l b_q^{(l)}$ and $B = \sum_{l=1}^{m_p} d_l$. Then (3.9) can be rewritten as

$$\sum_{l=1}^{m_p} f_l V_{p+l}^* + \sum_{q=p+m_p+1}^{M_p-1} e_q V_q{}^* = B \cdot \hat{V}_p . \tag{3.10}$$

Not all $f_l$ are zero since some $d_l \neq 0$ and all $a_{p+l} \neq 0$.

*Case* 1. $B = 0$. Theorem 3.2 implies the vectors $V_{p+1}^*, \ldots, V_{M_p-1}^*$ are independent for almost all $\mathbf{x}$. Thus, $B = 0 \Rightarrow$ all $f_l$ and $e_q$ are zero. This is a contradiction.

*Case* 2. $B \neq 0$. The definition of $W_q^{(n-1-p)}$ gives the identity

$$\sum_{q=p+m_p+1}^{M_p-1} e_q W_q^{(n-1-p)} + \sum_{l=1}^{m_p} f_l W_{p+l}^{(n-1-p)} \equiv B \left\{ W_p^{(n-1-p)} - (-1)^{n-1-p} \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)} \right\} \tag{3.11}$$

which is valid for $t < x_1$. But the LHS is already zero. Since $W_p^{(n-1-p)}(t) \equiv (-1)^{n-1-p} \Delta$ for $t < x_1$, (3.11) reduces to $\Delta = \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)}$. Theorem 3.3 says this is not an identity in $\mathbf{x}$ for $p \neq 0$ and, hence, it fails for almost all choices of $\mathbf{x}$. ▮

These results are summarized in Theorem 4.1 of Section 4.

## 4. Applications and Examples

Several consequences of Theorem 4.1 are derived which relate to the problems of interpolation of Hermite–Birkhoff data by functions with a specified derivative; to the problem of $E$ being poised; and to the problem of simple matrices. Two examples are discussed, one of which shows that the converse of Theorem 1.3 does not hold.

THEOREM 4.1. *Let $E$ satisfy* (SPC). *Let $p$ be a given integer, $1 \leqslant p \leqslant n-1$ and $g \in C^{(n-p)}[0, 1]$. For almost every* $\mathbf{x}$, *there is an $f \in Z(E, \mathbf{x})$ for which $f^{(p)} \equiv g$ iff $\langle g^{(q-p)}, W_q^{(n-q)} \rangle = 0$ for $q = p,..., n-1$.*

1. *Applications*

COROLLARY 4.1 (Birkhoff). *Let $p = n-1$. There is an $f \in Z(E, \mathbf{x})$ for which $f^{(n-1)} \equiv g$ iff $\langle g, W'_{n-1} \rangle = 0$. This statement holds for every* $\mathbf{x}$.

*Proof.* Corollary 3.1 shows that Theorem 3.2 holds everywhere if $p = n-1$. Also, Corollary 3.2 holds everywhere since $\{i: e_{i,n-1} = 1\} = \varnothing$ when $E$ satisfies (SPC). ▮

COROLLARY 4.2. *$E$ is poised at* $\mathbf{x}$ *iff $\int_0^1 W'_{n-1}(t)\, dt \neq 0$.*

*Proof.* $\int_0^1 W'_{n-1}(t)\, dt = \Delta$. ▮

COROLLARY 4.3 (Interpolation). *Let $E$, $p$, $g$ be given as in Theorem* 4.1. *Let $y_{ij}$ be data corresponding to $e_{ij} = 1$. For almost all* $\mathbf{x}$, *there is a function $f(x)$ for which $f^{(j)}(x_i) = y_{ij}$ when $e_{ij} = 1$ and*

$$f^{(p)} \equiv g \quad \text{iff} \quad \langle g^{(j-p)}, W_q^{(n-q)} \rangle = \sum_{j \leqslant q-1} y_{ij} \Delta_{ij}^{(q)}$$

*for $q = p,..., n-1$.*

*Proof.* Let $p(x)$ be the unique element of $\Pi_{n-1}$ satisfying $p^{(j)}(x_i) = y_{ij}$ when $e_{ij} = 1$. Suppose there is such a function $f$. Then $f - p \in Z(E, \mathbf{x})$ and $f^{(p)} - p^{(p)} \equiv g - p^{(p)} = h$. Thus,

$$0 = \langle h^{(j-p)}, W_q^{(n-q)} \rangle = \langle g^{(j-p)}, W_q^{(n-q)} \rangle - \langle p^{(p)}, W_q^{(n-q)} \rangle$$

for $q = p,..., n-1$. Now Lemma 2.1 gives

$$\begin{aligned} \langle p^{(p)}, W_q^{(n-q)} \rangle &= \sum_{j \leqslant q-1} p^{(j)}(x_i)\, \Delta_{ij}^{(q)} \\ &= \sum_{j \leqslant q-1} y_{ij} \Delta_{ij}^{(q)} \end{aligned}$$

and the implication is shown one way.

Suppose $\langle g^{(j-p)}, W_q^{(n-q)} \rangle = \sum_{j \leqslant q-1} y_{ij} \Delta_{ij}^{(q)}$. Then $\langle h^{(j-p)}, W_q^{(n-q)} \rangle = 0$ for $q = p,..., n-1$. Thus, there exists $f \in Z(E, \mathbf{x})$ such that $f^{(p)} \equiv h \equiv g - p^{(p)}$. The function $\hat{f} = f + p$ satisfies $\hat{f}^{(j)}(x_i) = y_{ij}$ when $e_{ij} = 1$ and $\hat{f}^{(p)} = f^{(p)} + p^{(p)} = g - p^{(p)} + p^{(p)} = g$. ▮

2. *Simple Matrices*

COROLLARY 4.4. *E is simple at* **x** *only if the function* $W'_{n-1}$ *is strictly of one sign.*

*Proof.* If $W'_{n-1}$ has a sign change then there is a strictly positive function $g$ such that $\langle g, W'_{n-1}\rangle = 0$. According to Corollary 4.1 there is an $f \in Z(E, \mathbf{x})$ such that $f^{(n-1)} \equiv g > 0$. Hence, $E$ cannot be simple. ∎

Consider again the example

$$E_1 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$$

of Section 1. It was shown there that $E_1$ is not simple for every **x**. This can also be shown using Corollary 4.4. Here, $\Delta_{1,0}^{(2)} = +1$, $\Delta_{2,1}^{(2)} = x_3 - x_1$ and $\Delta_{3,0}^{(2)} = -1$. Thus,

$$W_2'(t) = -(x_1 - t)_+ + (x_1 - x_3)(x_2 - t)_+^0 + (x_3 - t)_+$$

$$= \begin{cases} 0 & t < x_1 \\ t - x_1 & x_1 < t < x_2 \\ t - x_3 & x_2 < t < x_3 \\ 0 & x_3 < t \end{cases}.$$

This function always has a sign change at $t = x_2$. By Corollary 4.4 $E_1$ is not simple at any value of $(x_1, x_2, x_3) = \mathbf{x}$.

Now consider the following example.

$$E = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Lorentz and Zeller [5] have shown that this matrix is order-poised. The question arises: Is it simple for all **x**? Here, $n = 6$. Fix $x_1 = 0$, $x_2 = z$, $x_3 = 1$.

$$\Delta_{1,0}^{(5)} = \frac{z}{2} - \frac{z^2}{2}, \qquad \Delta_{2,4}^{(5)} = -\frac{1}{72}z^3 + \frac{1}{48}z^2 - \frac{1}{144}z,$$

$$\Delta_{1,1}^{(5)} = -\frac{1}{2}z^2 + \frac{1}{3}z - \frac{1}{12}, \qquad \Delta_{3,0}^{(5)} = \frac{z^2}{2} - \frac{z}{2},$$

$$\Delta_{2,1}^{(5)} = \frac{1}{12}, \qquad \Delta_{3,1}^{(5)} = \frac{z}{6} - \frac{z^2}{4}.$$

$$W_5'(t) = \begin{cases} \dfrac{t^3}{144}[2 + 6z^2 - 8z + (3z - 1379)\,t] & 0 < t < z \\ \dfrac{(t-1)^3}{6}\left[\dfrac{1}{8}(z^2 - z)(1 - t) + \dfrac{z}{6} - \dfrac{z^2}{4}\right] & z < t < 1 \end{cases}$$

A computation reveals the following.

(i) $0 < z < 1/3$ or $2/3 < z < 1$: $W_5'(t)$ has exactly one sign change and this occurs at $t = z$.

(ii) $1/3 < z < 1 - 1/\sqrt{3}$: $W_5'(t)$ has two sign changes occurring at $t = z$ and $t = -(2 + 6z^2 - 8z)/(3z - 3z^2)$.

(iii) $1/\sqrt{3} < z < 2/3$: $W_5'(t)$ has two sign changes occurring at $t = z$ and $t = 1 + (2/3)(2 - 3z)/(z - 1))$.

(iv) $1 - 1/\sqrt{3} < z < 1/\sqrt{3}$: $W_5'(t)$ has no sign changes.

Thus, if $z$ satisfies (i), (ii), or (iii), $E$ is not simple at the vector $\mathbf{x} = (0, z, 1)$. However, if $z$ satisfies (iv), $E$ is simple at $(0, z, 1)$. This example shows that the converse of Theorem 1.3 does not hold.

## 5. Proof of Theorem 3.3

The theorem to be proven is

THEOREM 3.3. *As a function of* $\mathbf{x}$, $\Delta = \sum_{i\in\Lambda_p} \Delta_{ip}^{(p)}$ *iff* $M_{p-1} = p$.

In the theorem, $\Delta$ is the VdM determinant of an incidence matrix $E$ and $\Lambda_p = \{i: e_{ip} = 1\}$. The concept of coalescing rows of a matrix $E$ and some lemmas on polynomial identities are needed for the proof. Once these are established the proof proceeds by cases depending on $k$ and $m_p$.

The coalescing of rows $i$, $i'$ of $E$ proceeds as follows. Let row $i$ have $t$ ones in it given by $e_{i,j_1} = e_{i,j_2} = \cdots = e_{i,j_t} = 1$. A new sequence $l_1, \ldots, l_t$ is defined by

$$\begin{aligned} &\text{(i)} \quad l_q \geqslant j_q \qquad q = 1,\ldots, t \\ &\text{(ii)} \quad e_{i',l_q} = 0 \qquad q = 1,\ldots, t \\ &\text{(iii)} \quad \sum_{q=1}^{t} (l_q - j_q) = \text{minimum over sequences satisfying (i) and (ii).} \end{aligned} \tag{5.1}$$

The coalesced matrix $E_{ii'}$ is formed by deleting rows $i$, $i'$ and replacing them with the single row $i^*$ defined by

$$e_{i^*j} = \begin{cases} 1, & e_{i'j} = 1 \quad \text{or} \quad j = l_q, \quad q = 1,\ldots, t; \\ 0, & \text{otherwise.} \end{cases}$$

As an example let

$$E = \begin{Vmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Then

$$E_{12} = \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad E_{23} = \begin{Vmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{Vmatrix}.$$

The following facts about $E_{ii'}$ will be needed.

$$\Delta_{ii'} = \frac{\partial^m}{\partial x_i^{\,m}} \Delta \Big|_{x_i = x_{i'}} \quad \text{where} \quad m = \sum_{q=1}^{t} (l_q - j_q). \tag{5.2}$$

$$E_{ii'} = E_{i'i}\,. \tag{5.3}$$

$$\text{If } E \text{ satisfies (PC) then } E_{ii'} \text{ also satisfies (PC).} \tag{5.4}$$

LEMMA 5.1. *Let $E$ satisfy* (PC) *with $k = 3$. Suppose one row consists entirely of zeros except for a single one. If that one does not occur in the initial position and $M_{p-1} \neq p$, then there is a real vector $(x_1, x_2, x_3)$ at which $E$ is not poised.*

*Proof.* This is a special case of a more general theorem of Lorentz and Zeller [5]. ∎

LEMMA 5.2. $\Delta(x_1, ..., x_k) \equiv \Delta(x_1 + t, ..., x_k + t)$ *for every $t$ and every* $\mathbf{x}$.

*Proof.* It is clear that $\Delta(x_1, ..., x_k) = 0$ iff $\Delta(x_1 + t, ..., x_k + t) = 0$. Thus, as polynomials in the variables $x_i$ they both have the same zero sets. Hence, they are identical. ∎

For $0 \leqslant q \leqslant n - 1$ and $e_{ij} = 1$ define the incidence matrix $E_{ij}^{(q)}$ by joining a $(k + 1)$ row with zeros everywhere except in position $q$ and then changing $e_{ij}$ to zero. Let $\theta = x_{k+1}$ and $\Delta_{ij}^{(q)}(\theta)$ be the corresponding VdM determinant. Note that $\Delta_{ij}^{(q)}(0) = \Delta_{ij}^{(q)}$.

LEMMA 5.3. $\Delta \equiv \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)}$ *iff* $\Delta \equiv \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)}(\theta)$ *for every $\theta$.*

*Proof.* The sufficiency is trivial. On the other hand, suppose $\Delta \equiv \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)}$. By Lemma 5.2,

$$\begin{aligned} \Delta(x_1, ..., x_k) &\equiv \Delta(x_1 - \theta, ..., x_k - \theta) \\ &\equiv \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)}(x_1 - \theta, ..., x_k - \theta, 0) \equiv \sum_{i \in \Lambda_p} \Delta_{ip}^{(p)}(x_1, ..., x_k, \theta). \quad \blacksquare \end{aligned}$$

*Proof of Theorem* 3.3

(1) *Sufficiency.* Suppose $M_{p-1} = p$. If $p = 0$ in which case $M_{-1} = 0$ then $\sum_{i \in \Lambda_p} \Delta_{i,0}^{(0)}$ is the expansion of $\Delta$ by cofactors along its last column. Hence, $\Delta \equiv \sum_{i \in \Lambda_p} \Delta_{i,0}^{(0)}$. Suppose $p > 0$. According to Theorem 1.1(ii) $E$ can be written as $E = E' \oplus E''$ where $E'$ is a $p$-incidence matrix and $E''$ is an $(n - p)$-incidence matrix. In this case the VdM matrix has the form

$$V = \begin{pmatrix} A & V' \\ V'' & 0 \end{pmatrix}$$

where $V'$, $V''$ are the VdM matrices of $E'$, $E''$, respectively. Thus, $\Delta = -\Delta'\Delta''$. By induction $\Delta'' = \sum_{i \in \Lambda_p} \Delta_{i,0}^{\prime\prime(0)}$ and it is easily checked that $\Delta'\Delta_{i,0}^{\prime\prime(0)} = \Delta_{i,p}^{(p)}$. Hence, sufficiency is shown.

(2) *Necessity.* The proof of necessity is divided into four cases. It is assumed that $p > 0$ throughout the discussion and that $E$ satisfies (PC) and that $M_{p-1} \geqslant p + 1$.

*Case* 1. $k = 2$, $m_p = 1$. Suppose $\Delta \equiv \Delta_{ip}^{(p)}(\theta)$. The matrix $E_{ip}^{(p)}$ satisfies the conditions of Lemma 5.1. Hence, it is not poised at some point $(x_1, x_2, \theta)$. But $E$ is unconditionally poised by Theorem 1.2. This is a contradiction.

*Case* 2. $k = m_p = 2$. Suppose $\Delta \equiv \Delta_{1p}^{(p)}(\theta) + \Delta_{2p}^{(p)}(\theta)$. Again it will be shown that under this hypothesis $\Delta$ is not unconditionally poised. Observe that $(d^q/d\theta^q)\, \Delta_{ip}^{(p)}(\theta) \equiv \Delta_{ip}^{(p+q)}(\theta)$. Let $q^* \geqslant p + 1$ be the first index for which $M_{q^*} = q^* + 2$. Differentiating $(q^* - p)$ times with respect to $\theta$ and remembering that $\Delta$ is a constant in $\theta$, one obtains $0 \equiv \Delta_{1p}^{(q^*)}(\theta) + \Delta_{2p}^{(q^*)}(\theta)$. By the choice of $q^*$ the two matrices $E_{1p}^{(q^*)}$ and $E_{2p}^{(q^*)}$ satisfy the conditions of Lemma 5.1. Thus, there is a choice of $\theta$ for which $\Delta_{1p}^{(q^*)}(\theta) = \Delta_{2p}^{(q^*)}(\theta) = 0$. Hence, there exist nontrivial polynomials $p_i(x)$ of degree less than $n$ satisfying $p_i(x) \in Z(E_{ip}^{(q)}, \mathbf{x})$. Construct an $(n - 1)$-incidence matrix $\tilde{E}$ from $E$ by changing $e_{1p}$ and $e_{2p}$ to zero and adding a row with a one in the $q^*$ position and zeros elsewhere. By the choice of $q^*$,

$$\tilde{E} = E_1 \oplus E_2 \oplus E_3 \qquad \left( E_2 = \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} \right)$$

where each $E_i$ is unconditionally poised. Hence, $\tilde{E}$ is an unconditionally poised $(n - 1)$ matrix. Each $p_i(x) \in Z(\tilde{E}, \mathbf{x})$. Thus, degree $p_i = n - 1$ and $p_i(x) \equiv d \cdot p_2(x)$ for a constant $d$. This in turn implies $p_1(x) \in Z(E_{i1}^{(q^*)}, \mathbf{x}) \cap Z(E_{i2}^{q^*}, \mathbf{x})$ which gives $p_1(x) \in Z(E, \mathbf{x})$. This is a contradiction.

*Case* 3. $k \geqslant 3$, $m_p = k$. In order to handle this case some further properties of the VdM determinant $\Delta$ are necessary. In particular, the degree of $\Delta$ in $x_i$ and the order of zero that $\Delta$ has at $x_i = x_i'$ are needed. Suppress row $i$ of the matrix $E$. Then the remaining matrix can be written as

$$E_1 \oplus E_2 \oplus \cdots \oplus E_{2r} \tag{5.5}$$

where the matrices with odd indices satisfy (PC) and those with even indices are zero matrices. If row $i$ has $t$ ones in it given by $e_{i,j_q} = 1$, $q = 1,..., t$, then the zero matrices of (5.5) will have a total of $t$ columns. These columns have labels $l_q^*$ in $E$ with each $l_q^* \geqslant j_q$. The degree of $\Delta$ as a polynomial in $x_i$ is $m^* = \sum_{q=1}^{t} (l_q^* - j_q)$. Also, $(\partial^{m^*}/\partial x_i^*)\Delta$ is the VdM determinant of the matrix $E^*$ obtained from (5.5) by putting a one in each of the columns of the even indexed matrices. Finally, the order of zero of $\Delta$ at $x_i = x_{i'}$ is the number $m \leqslant m^*$ defined by (5.1). For proofs of these statements the reader is referred to [3].

DEFINITION 5.1. Column $q$ of the incidence matrix $E$ is free in $E$ if $M_{q-1} = q$.

LEMMA 5.4. *Suppose* $m_p = k \geqslant 3$ *and* $\Delta \equiv \sum_{i \in A_p} \Delta_{ip}^{(p)}$. *If* $E_{2s+1}$ *is the matrix in* (5.5) *that contains the remainder of column* $p$, *then that column is free in* $E_{2s+1}$.

*Proof.* Without loss of generality take $i = 1$ in (5.4). Then $\Delta$ and each $\Delta_{ip}^{(p)}$ $i \neq 1$ will have degree $m^*$ in $x_1$. The degree of $\Delta_{ip}^{(p)}$ in $x_1$ will be $m^* - l_q^* - p < m^*$ where $l_{q-1}^* < p < l_q^*$. Lemma 5.4 can be assumed to hold for matrices with fewer than $k$ rows since by Case 2 it holds for two rows. Then

$$\frac{\partial^{m^*}}{\partial x_1^{m*}} \Delta \equiv \sum_{i \neq 1} \frac{\partial^{m^*}}{\partial x_1^{m*}} \Delta_{ip}^{(p)}.$$

But this is a representation of the VdM of $E^*$ along its $p$-th column. This is possible by induction iff the $p$-th column of $E^*$ is free in the submatrix $E_{2s+1}$. ∎

The next lemma shows that in some cases, if an identity of the type being discussed holds, then it carries over to the coalesced matrix.

LEMMA 5.5. $m_p = k$ *and* $\Delta \equiv \sum_{i \in A_p} \Delta_{ip}^{(p)}$. *Suppose* $E$ *has two rows (say rows* 1 *and* 2) *for which* $l_q < p$ *whenever* $j_q < p$ *in* (5.1). *Then the identity carries over to a similar one for the coalesced matrix* $E_{12}$.

*Proof.* Let $m = \sum_{q=1}^{t} (l_q - j_q)$ where row 1 of $E$ has $t$ ones in it corresponding to the index pairs $(1, j_q)$. $m$ is the number given by (5.1). Let $\tilde{\Delta}$ be the VdM determinant of $E_{12}$.

Then

$$\tilde{\Delta} = \frac{\partial^m}{\partial x_1{}^m} \Delta \Big|_{x_1=x_2}$$

according to (5.2). Also,

$$\tilde{\Delta}_{ip}^{(p)} = \frac{\partial^m}{\partial x_1{}^m} \Delta_{ip}^{(p)} \Big|_{x_1=x_2}$$

for $i = 3,..., k$. Differentiating the expression for $\Delta$ gives

$$\tilde{\Delta} = \frac{\partial^m}{\partial x_1{}^m} \Delta_{1p}^{(p)} \Big|_{x_1=x_2} + \frac{\partial^m}{\partial x_1{}^m} \Delta_{2p}^{(p)} \Big|^{+}_{x_1=x_2} + \sum_{i\geqslant 3} \tilde{\Delta}_{ip}^{(p)}.$$

Thus, it must be shown that

$$\frac{\partial^m}{\partial x_1{}^m} \Delta_{1p}^{(p)} \Big|_{x_1=x_2} + \frac{\partial^m}{\partial x_2{}^m} \Big|_{x_1=x_2} = \tilde{\Delta}_{1p}^{(p)}.$$

Let $y_1 ,..., y_{t'}$, be the column indices of the ones in row 2. Since $e_{1p} = e_{2p} = 1$, there are indices $s$, $s'$ for which $j_s = y_{s'} = p$. The determinants $\Delta_{1p}^{(p)}$, $\Delta_{2p}^{(p)}$ can be represented schematically by the sequences

$$(j_1 ,..., j_{s-1} , p^*, j_{s+1} ,..., j_t , y_1 ,..., y_{t'})$$

and

$$(j_1 ,..., j_t , y_1 ,..., y_{s'-1} , p^*, y_{s'+1} ,..., y_{t'}).$$

In this representation the indices $j_q$ represent rows of the determinant of the form

$$\left(\frac{x_1^{n-1-j_q}}{(n-1-j_q)}, \frac{x_1^{(n-2-j_q)}}{(n-2-j_q)} ,..., 1, 0,..., 0\right)$$

Similarly $y_q$ represents rows of the same form with $j_q$ and $x_1$ replaced by $y_q$ and $x_2$. The index $p^*$ represents the row $(0,..., 0, 1, 0,..., 0)$ with the one appearing in the $p$-th position. Formally, $(\partial^m/\partial x_1{}^m)\, \Delta_{2p}^{(p)}$ is a sum of determinants of the form

$$(j_1 + r_1 ,..., j_t + r_t , y_1 ,..., y_{s'-1} , p^*, y_{s'-1} ,..., y_{t'}) \tag{5.6}$$

with each $j_q + r_q < j_{q+1} + r_{q+1}$ and $\sum_{q=1}^{t} r_q = m$. Now, when $x_1$ is set equal to $x_2$, many of the forms (5.6) will be zero since they will have identical rows. Those that are not a priori zero must satisfy

$$r_q = l_q - j_q \qquad \text{for} \quad q = 1,..., s-1 \tag{5.7}$$

because of the assumption that $l_q < p$ whenever $j_q < p$. Also, they must satisfy

$$j_q + r_q \neq y_{q'} \qquad \text{for each } q \text{ and } q'. \tag{5.8}$$

Among all the forms (5.6) satisfying (5.7) and (5.8), the one with $r_q = l_q - j_q$ for each $q = 1,..., t$ gives $\tilde{\Delta}_{1p}^{(p)}$ when $x_1 = x_2$. The remaining terms have the form

$$(l_1 ,..., l_{s-1}, p, j_{s+1} + r_{s+1} ,..., j_t + r_t , y_1 ,..., y_{s'-1}, p^*, y_{s'+1} ,..., y_{t'}). \tag{5.9}$$

A similar analysis on $(\partial^m/\partial x_1{}^m)\, \Delta_{1p}^{(p)}$ shows that when $x_1 = x_2$ this quantity consists of determinants of the form

$$(l_1 ,..., l_{s-1}, p^*, j_{s+1} + r_{s+1} ,..., j_t + r_t , y_1 ,..., y_{s'-1}, p_1 y_{s'+1} ,..., y_{t'}). \tag{5.10}$$

These differ from those of (5.9) by having rows $p$, $p^*$ interchanged. Thus, they cancel when $(\partial^m/\partial x_1{}^m)\, \Delta_{1p}^{(p)} |_{x_1=x_2}$ is added to $(\partial^m/\partial x_1{}^m)\, \Delta_{2p}^{(p)} |_{x_1=x_2}$ and it has been shown that

$$\frac{\partial^m}{\partial x_1{}^m} \Delta_{1p}^{(p)} \bigg|_{x_1=x_2} + \frac{\partial^m}{\partial x_1{}^m} \Delta_{2p}^{(p)} \bigg|_{x_1=x_2} = \tilde{\Delta}_{1p}^{(p)}.$$

LEMMA 5.6. *Suppose $m_p = k \geqslant 3$ and there is some row $i$ for which column $p$ is free in the decomposition* (5.5). *Then $E$ satisfies the hypothesis of Lemma* 5.5.

*Proof.* Let column $p$ be free when row $i$ is suppressed. Then (5.5) can be written as $E_1 \oplus \cdots \oplus E'_{2s+1} \oplus E''_{2s+1} \oplus \cdots \oplus E_{2r}$. The remainder of column $p$ is the first column of $E''_{2s+1}$. Let $i'$, $i''$ be any two rows of $E$ except the given row $i$. Since the coalescing of row $i'$ and $i''$ depend only on their structure, the numbers $l_q$ may be determined by coalescing in the decomposition. If $j_q < p$ then $e_{i',j_q}$ lies in $E_{2c+1}$ for $c < s$ or in $E'_{2s+1}$. $l_q$ will lie in the same matrix. Hence, $l_q < p$.

These lemmas are used to show that the identity $\Delta = \sum_{i\in\Lambda_p} \Delta_{ip}^{(p)}$ does not hold in Case 3 (i.e., $m_p = k \geqslant 3$). In fact, if the identity holds, then Lemma 5.4 implies the remainder of column $p$ is free in the decomposition (5.5). But then Lemma 5.6 implies the hypotheses of Lemma 5.5 hold. Thus, the identity reduces to a similar one for the reduced matrix. By induction this must fail.

*Case* 4. $k \geqslant 3$ and $m_p < k$. Without loss of generality it may be assumed that $1 \in \Lambda_p$ and $k \notin \Lambda_p$. Let $m$ be the order of the zero of $\Delta$ at $x_1 = x_k$. For each $i \in \Lambda_p$, $i \neq 1$ $\Delta_{ip}^{(p)}$ has the same order zero at $x_1 = x_k$ as does $\Delta$. Thus, $\Delta_{1p}^{(p)}$ must have a zero of order $m$ at $x_1 = x_k$. The order of this zero is $m - (l_s - j_s)$ where $j_s = p$. Thus, $l_s = j_s$. But now the coalesced matrix $E_{1k}$ satisfies $M_{p-1} \geqslant p + 1$ and satisfies an identity of the same type as $\Delta$. The reduction can be continued until Case 1, 2, or 3 holds. This yields a contradiction. Thus, Theorem 3.3 is proven. ∎

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